

AUTOMORPHISMS OF BLOWUPS OF THREEFOLDS BEING FANO OR HAVING PICARD NUMBER 1

TUYEN TRUNG TRUONG

ABSTRACT. Let X_0 be a smooth projective threefold which is Fano or which has Picard number 1. Let $\pi : X \rightarrow X_0$ be a finite composition of blowups along smooth centers. We show that for "almost all" of such X , if $f \in \text{Aut}(X)$ then its first and second dynamical degrees are the same. We also construct many examples of finite blowups $X \rightarrow X_0$, on which any automorphism is of zero entropy.

The main idea is that because of the log-concavity of dynamical systems and the invariance of Chern classes under holomorphic automorphisms, there are some constraints on the nef cohomology classes.

We will also discuss a possible application of these results to a threefold constructed by Kenji Ueno.

1. INTRODUCTION

While there are many examples of compact complex surfaces having automorphisms of positive entropies (works of Cantat [10], Bedford-Kim [5][6][7], McMullen [27][28][29][30], Oguiso [32][33], Cantat-Dolgachev [11], Zhang [46], Diller [17], Déserti-Grivaux [16], Reschke [38],...), there are few interesting examples of manifolds of higher dimensions having automorphisms of positive entropies (Oguiso [34][35], Oguiso-Perroni [31],...). In particular, for the class of smooth rational threefolds, there are currently only two known examples of manifolds with primitive automorphisms of positive entropy (see [36, 13, 12]). Here a primitive automorphism, defined by D.-Q. Zhang [46], is one that has no non-trivial invariant fibrations over a base of dimension 1 or 2. For general properties on automorphism groups of compact Kähler manifolds, see the recent survey paper [18].

Then, it is natural to ask for what happens in dimension 3 and higher. For example, the following question was asked by Eric Bedford in 2011:

Question 1. Is there a finite composition of blowups at points or smooth curves $X \rightarrow \mathbb{P}^3$ starting from \mathbb{P}^3 and an automorphism $f : X \rightarrow X$ with positive entropy?

This paper aims to study Question 1 and some related questions. We give many evidences to that the answer to Question 1 is negative and to that the examples in [36, 13, 12] can not be obtained as smooth blowups of smooth threefolds having Picard number 1 or being Fano.

Our results and proofs are stated in terms of dynamical degrees, which we recall now. Let X be a smooth projective threefold. We denote by $\text{Pic}(X)$ the Picard group of X , $\text{Pic}_{\mathbb{Q}}(X) = \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\text{Pic}_{\mathbb{R}}(X) = \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\text{Nef}(X) \subset \text{Pic}_{\mathbb{R}}(X)$ be the cone of nef classes, which is the closure of the cone of ample classes.

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By Kleiman's criterion, a class in $Pic_{\mathbb{R}}(X)$ is nef iff it has non-negative intersection with every curve on X . For later use, we denote by $c_1(X)$ and $c_2(X)$ the first and second Chern classes of X . Let $f : X \rightarrow X$ be an automorphism. Then f preserves both $Pic(X)$ and $Nef(X)$. Let ω be an ample class on X . We define the first and second dynamical degrees of f as follows:

$$\begin{aligned}\lambda_1(f) &= \lim_{n \rightarrow \infty} [(f^n)^*(\omega) \cdot \omega^2]^{1/n}, \\ \lambda_2(f) &= \lim_{n \rightarrow \infty} [(f^n)^*(\omega^2) \cdot \omega]^{1/n}.\end{aligned}$$

Here are some properties of these dynamical degrees: $\lambda_1(f)^2 \geq \lambda_2(f) \geq 1$ and $\lambda_1(f^{-1}) = \lambda_2(f)$. For more on dynamical degrees see [21].

Entropy of f can be computed via dynamical degrees by Gromov-Yomdin's theorem [24, 42]: $h_{top}(f) = \log \max\{\lambda_1(f), \lambda_2(f)\}$. Hence, f has positive entropy iff $\lambda_1(f) > 1$.

Primitivity of f can also be detected from dynamical degrees via the following criterion (see [36]), which is a consequence of results in [19] and [20]: If $\lambda_1(f) \neq \lambda_2(f)$ then f is primitive.

The main idea behind all the results of this paper is that the existence of an automorphism f of positive entropy on X imposes some constraints on the cohomology groups of X . In fact, let $0 \neq \zeta \in Nef(X)$ be such that $f^*(\zeta) = \lambda_1(f)\zeta$ (the existence of such a class is guaranteed by Perron-Frobenius theorem). The differential df gives an isomorphism between the tangent bundle TX and its pullback $f^*(TX)$. Hence, from the properties of Chern classes we have $f^*c_1(X) = c_1(X)$ and $f^*c_2(X) = c_2(X)$. Since $\lambda_1(f) > 1$ and X has dimension 3, it follows that

$$\zeta^3 = \zeta^2 \cdot c_1(X) = \zeta \cdot c_1(X)^2 = 0.$$

In fact, stronger constraints are satisfied.

Theorem 1. *Let X be a projective manifold of dimension 3 and $f : X \rightarrow X$ an automorphism.*

- 1) *If f has positive entropy, there is a nef class ζ which is not in $\mathbb{R}.Pic_{\mathbb{Q}}(X)$ such that $\zeta^2 = 0$, $\zeta \cdot c_1(X)^2 = 0$ and $\zeta \cdot c_2(X) = 0$.*
- 2) *If $\lambda_1(f) \neq \lambda_2(f)$, there is a nef class ζ which is not in $\mathbb{R}.Pic_{\mathbb{Q}}(X)$ such that $\zeta^2 = 0$, $\zeta \cdot c_1(X) = 0$ and $\zeta \cdot c_2(X) = 0$.*

Here we comment on the condition $\zeta^2 = 0$. If X has dimension 2, then this condition is one homogeneous equation in m variables (here, m is the Picard number of X) and hence is very easily satisfied. In contrast, when X has dimension 3 or bigger, then the condition $\zeta^2 = 0$ is a system of $p \geq m$ homogeneous equations in the m variables (here p is the dimension of $\bigwedge^2 Pic_{\mathbb{R}}(X)$), and hence is more difficult to be satisfied. This is a heuristic argument for why it is difficult to find automorphisms of positive entropy in dimension 3 or larger.

Based on Theorem 1, we state some conditions on nef cohomology classes.

Condition 2. *Let X be a smooth projective threefold.*

- 1) *Condition A: We say that X satisfies Condition A if whenever $\zeta \in Nef(X)$ is such that $\zeta^2 = 0$, $\zeta \cdot c_1(X)^2 \geq 0$ and $\zeta \cdot c_2(X) \leq 0$, then $\zeta \in \mathbb{R}.Pic_{\mathbb{Q}}(X)$.*
- 2) *Condition B: We say that X satisfies Condition B if whenever $\zeta \in Nef(X)$ is such that $\zeta^2 = 0$, $\zeta \cdot c_1(X) = 0$ and $\zeta \cdot c_2(X) \leq 0$, then $\zeta \in \mathbb{R}.Pic_{\mathbb{Q}}(X)$.*

By Theorem 1, if X satisfies Condition A then any automorphism on X has zero entropy, and if X satisfies Condition B then for any automorphism f of X we have $\lambda_1(f) = \lambda_2(f)$. While requiring more than the assumptions in part 1) of Theorem 1, Condition A is very suitable for inductive arguments. A similar comment applies for Condition B.

Now we are ready to state the main results of this paper. The first result is for blowups of some special configurations of \mathbb{P}^3 .

Theorem 3. *Let p_1, \dots, p_n be distinct points in $X_0 = \mathbb{P}^3$ such that any 4 points of them do not belong to the same hyperplane. Let $C_{i,j}$ be the line connecting the points p_i and p_j . Let $\pi_1 : X_1 \rightarrow \mathbb{P}^3$ be the blowup at p_1, \dots, p_n . Let $D_{i,j} \subset X_1$ be the strict transforms of $C_{i,j}$, and $\pi_2 : X_2 \rightarrow X_1$ be the blowup at $D_{i,j}$. Then any automorphism of X_2 has zero entropy.*

Remark. Igor Dolgachev and Yuri Prokhorov informed us that in the special cases where $4 \leq n \leq 7$, then the automorphism group of X_2 in Theorem 3 is finite. The conclusion of Theorem 3 can be proved for the blowups of more general configurations in \mathbb{P}^3 . However, since the statements of these generalizations are a bit complicated, we refer to Section 4 for more details.

The next two main results of the paper are for threefolds having Picard number 1 or satisfying a special property on the second Chern class. We recall that a class ζ on X is movable if there is a smooth blowup $\pi : Z \rightarrow X$ such that ζ is the pushforward of some nef class on Z .

Theorem 4. *Let X_0 be a threefold with Picard number 1. Let $C_1, \dots, C_t \subset X_0$ be smooth curves which are pairwise disjoint. Let $p_1, \dots, p_s \in X_0$ be distinct points, which are allowed to belong to the curves C_1, \dots, C_t . Let $\pi_1 : X_1 \rightarrow X_0$ be the blowup at p_1, \dots, p_s , and $\pi_2 : X_2 \rightarrow X_1$ the blowup at C_1, \dots, C_t . Then X_2 satisfies Properties A and B.*

We note that in general Theorem 4 does not hold for threefolds X_0 with Picard number ≥ 2 (for example when $X_0 = \mathbb{P}^2 \times \mathbb{P}^1$). However, the theorems below may still hold for those manifolds. See Section 4 for more details.

Theorem 5. *Let X_0 be a smooth projective threefold such that $c_2(X_0) \cdot \zeta > 0$ for all non-zero movable $\zeta \in NS_{\mathbb{R}}(X_0)$. Let $p_1, \dots, p_n \in X_0$ be distinct points. Let $\pi_1 : X_1 \rightarrow X_0$ be the blowup of X_0 at p_1, \dots, p_n . Let $D_1, \dots, D_m \subset X_1$ be disjoint smooth curves, and $\pi_2 : X_2 \rightarrow X_1$ the blowup at D_1, \dots, D_m .*

1) X_2 satisfies Condition B.

2) Assume moreover that for any j , then $c_1(X_1) \cdot D_j \leq 2g_j - 2$, where g_j is the genus of D_j . Then X_2 satisfies Condition A.

Theorem 5 applies for $X_0 = \mathbb{P}^3$ or $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. It also applies for complete intersection threefolds in \mathbb{P}^N . See Section 4 for more details. We note that here the images in X_0 of D_1, \dots, D_m may be singular and intersect with each other, hence Theorem 5 is not covered by Theorem 4 even in the case $X_0 = \mathbb{P}^3$.

Finally, we state several results which are purely inductive in nature, which can be applied to blowups of Fano threefolds as well. Here we recall that a threefold is Fano if $c_1(X)$ is ample.

We start with the case of point blowups.

Theorem 6. *Let Y be a smooth projective threefold satisfying one of the Conditions A and B. Let $\pi : X \rightarrow Y$ be the blowup at a point. Then X satisfies the same Condition.*

Next, we consider the case of curve blowups.

Theorem 7. *Let Y be a smooth projective threefold satisfying Condition A or B. Let $\pi : X \rightarrow Y$ be the blowup at a smooth curve $C \subset Y$. Let g be the genus of C , and define $\gamma = c_1(Y).C + 2g - 2$. Then X also satisfies the same Condition, if one of the following cases happens.*

- 1) $c_1(Y).C$ is an odd number and the normal vector bundle $N_{C/Y}$ is decomposable. The latter means that $N_{C/Y}$ is the direct sum of two line bundles over C .
- 2) $\gamma < 0$ and C is not the only effective curve in its cohomology class.
- 3) There is an irreducible hypersurface $S \subset Y$ such that $2\kappa < \mu\gamma$. Here $\kappa = S.C$ and μ is the multiplicity of C in S .

We note that in 1) of Theorem 7, the condition that $N_{C/Y}$ is decomposable may be easily to satisfy. For example, if C is a smooth rational curve, then $N_{C/Y}$ is always decomposable by a result of Grothendieck, even if C does not move in Y .

Theorem 8. *Let Y be a smooth projective threefold satisfying Condition B. Let $\pi : X \rightarrow Y$ be the blowup at a smooth curve $C \subset Y$. Let g be the genus of C . If $c_1(Y).C \neq 2g - 2$, then X also satisfies Condition B.*

Hence, we conclude that if X_0 is a smooth threefold which is Fano or has Picard number 1, then for almost every $X \rightarrow X_0$ a finite composition of points or smooth curves, every automorphism f on X has $\lambda_1(f) = \lambda_2(f)$. This is a strong indication that probably all automorphisms on such manifolds are not primitive, i.e. has invariant fibrations over a base of dimension 1 or 2.

In Section 5 we will discuss possible application of the above results to the Ueno's threefold considered in [41]. In Section 4, we will give various examples illustrating the above results.

Remark. The general case of compact Kähler threefolds can be similarly treated, by replacing the Neron-Severi group by the $(1, 1)$ cohomology group. After the appearance of a first version of this paper (see [40]), some generalizations to higher dimensions have been given in [2] and [39].

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2. PRELIMINARIES ON NEF CLASSES AND BLOWUPS

2.1. Kähler, nef and psef classes, and effective varieties. Let X be a compact Kähler manifold. Let $\eta \in H^{1,1}(X)$. We say that η is Kähler if it can be represented by a Kähler $(1, 1)$ form. We say that η is nef if it is a limit of a sequence of Kähler classes. We say that η is psef if it can be represented by a positive closed $(1, 1)$ current. A class $\xi \in H^{p,p}(X)$ is an effective variety if there are irreducible varieties

C_1, \dots, C_t of codimension p in X and non-negative real numbers a_1, \dots, a_t so that ξ is represented by $\sum_i a_i C_i$.

Demailly and Paun [15] gave a characterization of Kähler and nef classes, which in the case of projective manifolds is summarized as follows:

Theorem 9. *Let X be a projective manifold with a Kähler $(1,1)$ form ω . A class $\eta \in H^{1,1}(X)$ is Kähler if and only for any irreducible subvariety $V \subset X$ then $\int_V \eta^{\dim(V)} > 0$. A class $\eta \in H^{1,1}(X)$ is nef if and only for any irreducible subvariety $V \subset X$ then $\int_V \eta^{\dim(V)-j} \wedge \omega^j \geq 0$ for all $0 \leq j \leq \dim(V)$.*

Nef classes are preserved under pullback by holomorphic maps.

Lemma 1. *Let $\pi : X \rightarrow Y$ be a holomorphic map between compact Kähler manifolds. Then $\pi^*(H_{nef}^{1,1}(X)) \subset H_{nef}^{1,1}(Y)$.*

Proof. Since nef classes are in the closure of Kähler classes, it suffices to show that if η is a Kähler class then $\pi^*(\eta)$ is nef. Let φ be a Kähler $(1,1)$ form representing η . Then $\pi^*(\varphi)$ is a positive smooth $(1,1)$ form. Let ω_X be a Kähler $(1,1)$ form on X . Then $\pi^*(\eta)$ is represented as a limit of the following Kähler classes

$$\pi^*(\varphi) + \frac{1}{n} \omega_X,$$

and hence is nef. \square

Remark: Similarly, it can be shown that psef classes are preserved under pushforward by holomorphic maps. However, nef classes may not be preserved under pushforwards, even when the map is a blowup.

2.2. Blowup of a projective 3-manifold at a point. Let $\pi : X \rightarrow Y$ be the blowup of a projective 3-manifold at a point p . Let $E = \mathbb{P}^2$ be the exceptional divisor and let $L \subset E$ be a line. Then $H^{1,1}(X)$ is generated by $\pi^*(H^{1,1}(Y))$ and E , and $H^{2,2}(X)$ is generated by $\pi^*(H^{2,2}(Y))$ and L . The intersection product on the cohomology of X is given by

$$\begin{aligned} \pi^*(\xi).E &= 0, \quad E.E = -L, \\ \pi^*(\xi).L &= 0, \quad E.L = -1. \end{aligned}$$

The first and second Chern classes of X can be computed by (see e.g. Section 6, Chapter 4 in the book of Griffiths-Harris [25])

$$\begin{aligned} c_1(X) &= \pi^*(c_1(Y)) - 2E, \\ c_2(X) &= \pi^*(c_2(Y)). \end{aligned}$$

The following result concerns the relations between cycles on X and Y .

Lemma 2. *For any effective curve $V \subset Y$, there is an effective curve $\tilde{V} \subset X$ so that $\pi_*(\tilde{V}) = V$ and $\tilde{V}.E \geq 0$.*

Proof. It suffices to consider the case when V is an irreducible curve. We can choose \tilde{V} to be the strict transform of V . Then $\pi_*(\tilde{V}) = V$, and \tilde{V} is not contained in E . Therefore $\tilde{V}.E \geq 0$. \square

We end this subsection showing that nef classes are preserved under pushforward by point-blowups.

Lemma 3. *Let $\eta \in H_{nef}^{1,1}(X)$. Then $\pi_*(\eta) \in H_{nef}^{1,1}(Y)$.*

Proof. It suffices to prove the conclusion when η is a Kähler class. Let φ be a Kähler $(1,1)$ form representing η . Then $\pi_*(\varphi)$ is a positive closed $(1,1)$ current, which is smooth on $X - p$.

Let ω_Y be a Kähler $(1,1)$ form on Y . To show that $\pi_*(\eta)$ is a nef class, by Theorem 9 it suffices to show that for any irreducible variety $V \subset Y$ then $\pi_*(\eta)^{\dim(V)-j} \cdot V \cdot \omega_Y^j \geq 0$ for $0 \leq j \leq \dim(V)$. We let $[V]$ be the current of integration on V . Then by the results in Section 4, Chapter 3 in the book of Demailly [14], the current $\pi_*(\varphi)^{\dim(V)-j} \wedge [V] \wedge \omega_Y^j$ is well-defined and is a positive measure, whose mass equals to $\pi_*(\eta)^{\dim(V)-j} \cdot V \cdot \omega_Y^j$. Thus the latter quantity is non-negative. \square

2.3. Blowup of a projective 3-manifold along a smooth curve. Let $\pi : X \rightarrow Y$ be the blowup of a projective 3-manifold along a smooth curve $C \subset Y$. Let g be the genus of C . Let F be the exceptional divisor and let M be a fiber of the projection $F \rightarrow C$. We can identify F with the projective bundle $\mathbb{P}(\mathcal{E}) \rightarrow C$, where $\mathcal{E} = N_{C/Y} \rightarrow C$ is the normal vector bundle of C in Y .

Then $H^{1,1}(X)$ is generated by $\pi^*(H^{1,1}(Y))$ and F , and $H^{2,2}(X)$ is generated by $\pi^*(H^{2,2}(Y))$ and M . The intersection between F and M is $F \cdot M = -1$. The first and second Chern classes of X can be computed as follows:

$$\begin{aligned} c_1(X) &= \pi^*(c_1(Y)) - F, \\ c_2(X) &= \pi^*(c_2(Y) + C) - \pi^*c_1(Y) \cdot F. \end{aligned}$$

Let $[F] \rightarrow X$ be the line bundle of F in X , and denote by $e = [F]|_F$. Then (see e.g. Section 6, Chapter 4 in the book of Griffiths - Harris [25]) in F we have the equalities

$$e \cdot M = -1, \quad e \cdot e = -c_1(\mathcal{E}).$$

From the SES of vector bundles on C

$$0 \rightarrow T_C \rightarrow T_Y|_C \rightarrow \mathcal{E} \rightarrow 0,$$

it follows by the additivity of first Chern classes that

$$c_1(\mathcal{E}) = c_1(T_Y) \cdot C - c_1(T_C) = c_1(Y) \cdot C + 2g - 2.$$

We define

$$\gamma := c_1(Y) \cdot C + 2g - 2.$$

Since $F \rightarrow C$ is a ruled surface (i.e. its fibers are projective lines \mathbb{P}^1), there is a canonical section C_0 which is the image of a holomorphic map $\sigma_0 : C \rightarrow F$ (see e.g. Section 2, Chapter 5 in Hartshorne's book [26]). Therefore C_0 is an effective curve in F . Such a C_0 has intersection 1 with a fiber M .

We will return to the canonical section C_0 at the end of this subsection. For now, we however work in a more general assumption on C_0 , for using later. That is, we consider an effective curve $C_0 \subset F$ with the following properties

$$\begin{aligned} C_0 \cdot C_0 &= \tau, \\ C_0 \cdot M &= \mu > 0, \\ M \cdot M &= 0. \end{aligned}$$

Any divisor on F is numerically equivalent to a linear combination of C_0 and M . We now show the following

Lemma 4. a)

$$(2.1) \quad F.C_0 = \frac{1}{2}(\gamma\mu - \frac{\tau}{\mu}).$$

b)

$$F.F = -\frac{1}{\mu}C_0 + \frac{1}{2}(\frac{\tau}{\mu^2} + \gamma)M.$$

c) $\pi_*(F.F) = -C$.

Proof. a) In fact, we have

$$F.C_0 = [F]|_{C_0} = [F]|_F.C_0 = e.C_0,$$

here the two expressions on the RHS are computed in F . On F , numerically we can write $e = aC_0 + bM$. Then from $-1 = e.M = (aC_0 + bM).M = a\mu$, we get $a = -1/\mu$. Substitute this into $e.e = -\gamma$ we obtain

$$-\gamma = e.e = (\frac{1}{\mu}C_0 - bM).(\frac{1}{\mu}C_0 - bM) = \frac{\tau}{\mu^2} - 2b,$$

which implies that

$$b = \frac{1}{2}(\frac{\tau}{\mu^2} + \gamma).$$

Therefore

$$e = \frac{-1}{\mu}C_0 + \frac{1}{2}(\frac{\tau}{\mu^2} + \gamma)M.$$

Thus

$$\begin{aligned} F.C_0 &= e.C_0 = [\frac{-1}{\mu}C_0 + \frac{1}{2}(\frac{\tau}{\mu^2} + \gamma)M]C_0 \\ &= \frac{-\tau}{\mu} + \frac{1}{2}(\frac{\tau}{\mu} + \gamma\mu) \\ &= \frac{1}{2}(-\frac{\tau}{\mu} + \gamma\mu). \end{aligned}$$

b) From the formula for e in the proof of a) it is not difficult to arrive at the proof of b).

c) Since $C_0.M = \mu$, it follows that $\pi_*(C_0) = \mu C$. Then from b) we obtain c). \square

We end this subsection commenting on conditions 2) and 3) of Theorem ?? . By Proposition 2.8 in Chapter 5 of [26], there is a line bundle $\mathcal{M} \rightarrow C$ so that the vector bundle $\mathcal{E}' = \mathcal{E} \otimes \mathcal{M}$ is normalized in the following sense: $H^0(\mathcal{E}') \neq 0$ but for all line bundle $\mathcal{L} \rightarrow C$ with $c_1(\mathcal{L}) < 0$ then $H^0(\mathcal{E}' \otimes \mathcal{L}) = 0$. A canonical section $C_0 \subset F$ can be associated to such a normalized \mathcal{E}' . The intersection between C_0 and M is 1. Moreover, the number

$$\tau_0 = C_0.C_0 = c_1(\mathcal{E}') = c_1(\mathcal{E}) + 2c_1(\mathcal{M}),$$

is an invariant of F .

We end this section with some further properties of a ruled surface.

Lemma 5. Assume that the invariant τ_0 of F is non-negative. Then

a) For any effective curve $V \subset F$ we have $V.V \geq 0$.

b) If moreover $\gamma < 0$ then for any non-zero effective curve $V \subset F$ we have $F.V < 0$.

Proof. a) It suffices to prove for the case V is an irreducible curve. Numerically, we write $V = aC_0 + bM$. If $V = C_0$ then $V.V = \tau_0 \geq 0$. If $V = M$ then $V.V = 0$. Hence we may assume that $V \neq C_0, M$.

We consider two cases:

Case 1: $\tau_0 = 0$. By Proposition 2.20 in Chapter 5 in [26], we have $a > 0$ and $b \geq 0$. Therefore

$$V.V = a^2\tau_0 + 2ab \geq 0.$$

Case 2: $\tau_0 > 0$. By Proposition 2.21 in Chapter 5 in [26], there are two subcases:

Subcase 2.1: $a = 1, b \geq 0$. Then

$$V.V = \tau_0 + 2b \geq 0.$$

Subcase 2.2: $a \geq 2, b \geq -a\tau_0/2$. Then

$$V.V = a^2\tau_0 + 2ab \geq a^2\tau_0 + 2a(-a\tau_0/2) = 0.$$

b) It suffices to prove for the case V is an irreducible curve. If $V = M$ then $F.M = -1 < 0$. If $V = C_0$ then by Lemma 4 with $\tau = \tau_0 \geq 0$ and $\mu = 1$

$$F.C_0 = \frac{1}{2}(\gamma - \tau_0) \leq \frac{1}{2}\gamma < 0$$

because $\gamma < 0$. Therefore we may assume that $V \neq C_0, M$, and then proceed as in the proof of a). \square

3. PROOFS OF THE MAIN RESULTS

We make use of the following result (see e.g. [44] and [3]).

Lemma 6. *Let X be a smooth projective threefold. Let $f : X \rightarrow X$ be an automorphism. If $\lambda_1(f) > 1$, then $\lambda_1(f)$ is irrational.*

For the convenience of the readers, we reproduce the proof of this Lemma here.

Proof. Let A be the matrix of $f^* : NS_{\mathbb{R}}(X) \rightarrow NS_{\mathbb{R}}(X)$, then A is an integer matrix, and $\lambda_1(f)$ is a real eigenvalue of A . Moreover, A is invertible and its inverse A^{-1} is the matrix of the map $(f^{-1})^* : NS_{\mathbb{R}}(X) \rightarrow NS_{\mathbb{R}}(X)$ hence is also an integer matrix. Therefore $\det(A) = \pm 1$. Thus the characteristic polynomial $P(x)$ of A is a monic polynomial of integer coefficients and $P(0) = \pm 1$. Assume that $\lambda_1(f)$ is a rational number. Since $\lambda_1(f)$ is an algebraic integer, it follows that $\lambda_1(f)$ must be an integer. Then we can write $P(x) = (x - \lambda_1(f))Q(x)$, here $Q(x)$ is a polynomial of integer coefficients. If $\lambda_1(f) > 1$ we get a contradiction $\pm 1 = P(0) = -\lambda_1(f)Q(0)$ \square

Now we give the proofs of the main results.

Proof of Theorem 1. 1) Since f^* preserves the cone $Nef(X)$, by a Perron-Frobenius type theorem, there is a non-zero nef class η so that $f^*(\eta) = \lambda_1(f)\eta$. Similarly, there is a non-zero nef class η_- so that $(f^{-1})^*(\eta_-) = \lambda_1(f^{-1})\eta_-$.

Assume that $\lambda_1(f) > 1$. By the log-concavity of dynamical degrees, we also have $\lambda_1(f^{-1}) > 1$. By Lemma 6, both $\lambda_1(f)$ and $\lambda_1(f^{-1})$ are irrational. Hence, both ζ and ζ_- are not in $\mathbb{R}.NS_{\mathbb{Q}}(X)$. It is easy to see that

$$\begin{aligned} \zeta.c_1(X)^2 &= \zeta.c_2(X) = 0, \\ \zeta_-.c_1(X)^2 &= \zeta_-.c_2(X) = 0. \end{aligned}$$

To prove 1) it suffices to show that either $\zeta^2 = 0$ or $\zeta_-^2 = 0$. Assume otherwise. From $\zeta^2 \neq 0$ and $f^*(\zeta^2) = \lambda_1(f)^2 \zeta^2$, we have

$$\lambda_1(f)^2 \leq \lambda_2(f) = \lambda_1(f^{-1}).$$

Similarly, from $\zeta_-^2 \neq 0$, we have

$$\lambda_1(f^{-1}) \geq \lambda_1(f)^2.$$

Combining these two inequalities, we conclude that $\lambda_1(f) \geq \lambda_1(f)^4$, which contradicts to $\lambda_1(f) > 1$. This completes the proof of 1).

2) The proof of 2) is similar. \square

Proof of Theorem 3. 1) For the proof, it suffices to show that for any non-zero nef ζ on $X = X_2$ then either $\zeta.c_1(X)^2 \neq 0$ or $\zeta.c_2(X) \neq 0$.

We let E_1, \dots, E_n be the exceptional divisors of the blowup $\pi_1 : X_1 \rightarrow X_0 = \mathbb{P}^3$. Let $F_{i,j}$ be the exceptional divisors of the blowup $\pi_2 : X = X_2 \rightarrow X_1$. Then we can write

$$\begin{aligned} \zeta &= \pi_2^*(\xi) - \sum_{i < j} \alpha_{i,j} F_{i,j}, \\ \xi &= \pi_1^*(u) - \sum_l \beta_l E_l. \end{aligned}$$

Here u is nef on \mathbb{P}^3 and $\alpha_{i,j}, \beta_l \geq 0$.

For the proof of 1), it then suffices to show that $\deg(u) = 0$. From

$$c_2(X) = \pi_2^* c_2(X_1) + \sum_{i < j} \pi_2^* D_{i,j} - \sum_{i < j} \pi_2^* c_1(X_1) \cdot F_{i,j},$$

and the fact that $c_1(X_1) \cdot D_{i,j} = 0$, the condition $\zeta.c_2(X) = 0$ becomes $\xi.c_2(X_1) + \sum_{i < j} \xi \cdot D_{i,j} = 0$. Since $c_2(X_1) = \pi_1^*(c_2(\mathbb{P}^3))$, it follows that $\xi.c_2(X_1) = 16 \deg(u)$. We also have that $\xi \cdot D_{i,j} = \deg(u) - \beta_i - \beta_j$ for every $i < j$. Therefore, we obtain

$$\begin{aligned} 6 \deg(u) &= - \sum_{i < j} \xi \cdot D_{i,j}, \\ (6 + \frac{n(n-1)}{2}) \deg(u) &= (n-1) \sum_l \beta_l. \end{aligned}$$

From the condition $\zeta.c_1(X)^2 = 0$, we obtain

$$\begin{aligned} 0 &= \zeta.c_1(X)^2 = (\pi_2^*(\xi) - \sum_{i < j} \alpha_{i,j} F_{i,j}) \cdot (\pi_2^* c_1(X_1)^2 - 2 \sum_{i < j} \pi_2^* c_1(X_1) \cdot F_{i,j} + \sum_{i < j} F_{i,j}^2) \\ &= \xi.c_1(X_1)^2 - \sum_{i < j} \xi \cdot D_{i,j} - 2 \sum_{i < j} \alpha_{i,j} c_1(X_1) \cdot D_{i,j} + \sum_{i < j} \alpha_{i,j} (c_1(X_1) \cdot D_{i,j} + 2g_{i,j} - 2) \\ &= 22 \deg(u) - 4 \sum_l \beta_l + \sum_{i < j} \alpha_{i,j} (2g_{i,j} - 2 - c_1(X_1) \cdot D_{i,j}) \\ &= 22 \deg(u) - 4 \sum_l \beta_l - 2 \sum_{i < j} \alpha_{i,j}. \end{aligned}$$

In the above, $g_{i,j} = 0$ is the genus of $C_{i,j}$, and $c_1(X_1) \cdot D_{i,j} = 0$ for all $i < j$. In particular, we obtain

$$(3.1) \quad \frac{11}{2} \deg(u) \geq \sum_l \beta_l = (\frac{6}{n-1} + \frac{n}{2}) \deg(u).$$

From the above inequality, we will finish showing that $\deg(u) = 0$. We consider several cases:

Case 1: $n \geq 10$. From Equation (3.1), it follows immediately that $\deg(u) = 0$ as wanted.

Case 2: $6 \leq n \leq 9$. In this case, for each 6 points p_{i_1}, \dots, p_{i_6} among n points p_1, \dots, p_n , there is a unique rational normal curve $C \subset \mathbb{P}^3$ of degree 3 passing through the 6 chosen points. Let $D \subset X_1$ be the strict transform of C . Then D is different from the curves $D_{i,j}$. Therefore $\pi_2^* D$ is an effective curve, and hence

$$3 \deg(u) - \sum_{l=1}^6 \beta_{i_l} \geq \xi \cdot D = \zeta \cdot \pi_2^*(D) \geq 0.$$

Summing over all such choices of p_{i_1}, \dots, p_{i_n} we find that

$$\frac{n}{2} \deg(u) \geq \sum_l \beta_l.$$

Combining this with

$$\sum_l \beta_l = \left(\frac{6}{n-1} + \frac{n}{2}\right) \deg(u),$$

we obtain $\deg(u) = 0$.

Case 3: $n = 4, 5$. In this case, we use rational normal curves to obtain

$$\frac{n}{3} \deg(u) \geq \sum_l \beta_l.$$

Combining this with

$$\sum_l \beta_l = \left(\frac{6}{n-1} + \frac{n}{2}\right) \deg(u),$$

we obtain $\deg(u) = 0$.

Case 4: $n = 1, 2, 3$. In this case we have $n \deg(u) \geq \sum_l \beta_l$. Combining this with

$$\sum_l \beta_l = \left(\frac{6}{n-1} + \frac{n}{2}\right) \deg(u),$$

we obtain $\deg(u) = 0$. □

Proof. (Proof of Theorem 4)

Let $\pi'_1 : X'_1 \rightarrow X_0$ be the blowup at the C_1, \dots, C_t . Let F_1, \dots, F_t be the exceptional divisors. Let $M_j = (\pi'_1)^{-1}(p_j)$ be the preimages of the points p_j ($j = 1, \dots, n$). These are smooth rational curves, and are among the fibers of the maps $F_1 \rightarrow C_1, \dots, F_t \rightarrow C_t$. Let $\pi'_2 : X'_2 \rightarrow X'_1$ be the blowup at the curves M_j . Then X_2 is isomorphic to X'_2 .

Fixed a number j . Let i be such that $M_j \subset F_i$. Using that

$$c_1(N_{M_j/X'_1}) = c_1(N_{M_j/F_i}) + c_1(N_{F_i/X'_1}|_{M_j}) = 0 + (-1) = -1,$$

we find that

$$c_1(X'_1) \cdot M_j = 1$$

is an odd number. Therefore, using either part 1) or part 3) of Theorem 7, for the proof of Theorem 5 it suffices to show that X'_1 satisfies both Conditions A and B. To this end, we only need to show that if $\zeta \in \text{Nef}(X'_1)$ is such that $\zeta^2 = 0$ then $\zeta \in \mathbb{R} \cdot \text{NS}_{\mathbb{Q}}(X'_1)$.

Let $H \in NS_{\mathbb{Q}}(X_0)$ be an ample divisor. Since X_0 has Picard number 1, we can write

$$\zeta = a(\pi'_1)^*(H) - \sum_j \alpha_j F_j,$$

where $a, \alpha_1, \dots, \alpha_t \geq 0$. If $a = 0$, then from the fact that ζ is nef, we have $\alpha_1 = \dots = \alpha_t = 0$. Therefore, we may assume that $\alpha > 0$, and after dividing by α we may assume that $\alpha = 1$. Then, for the proof of the theorem, it suffices to show that all the numbers $\alpha_1, \dots, \alpha_t$ are in \mathbb{Q} .

Since the curves C_j are pairwise disjoint, for any $i = 1, \dots, t$ we have

$$\begin{aligned} 0 &= \zeta^2.F_i = ((\pi'_1)^*H - \sum_j \alpha_j F_j)^2.F_i \\ &= (\pi'_1)^*H^2.F_i - 2\alpha_i(\pi'_1)^*H.F_i^2 + \alpha_i^2 F_i^3 \\ &= 2\alpha_i H.C_i - \alpha_i^2 (c_1(X_0).C_i + 2g_i - 2), \end{aligned}$$

here g_i is the genus of C_i . We note that $H.C_i$ is a positive rational number. Hence, either $\alpha_i = 0$, or

$$\alpha_i = 2H.C_i / (c_1(X_0).C_i + 2g_i - 2).$$

In both cases, α_i are rational numbers as wanted. \square

Proof of Theorem 5. 1) Let ζ be a nef class on X_2 such that $\zeta^2 = 0$, $\zeta.c_1(X) = 0$ and $\zeta.c_1(X_2)^2 \leq 0$. We need to show that $\zeta \in \mathbb{R}.H_{alg}^2(X_2, \mathbb{Q})$. More strongly, we will show that ζ must be 0.

Let us denote by F_j the exceptional divisor over D_j of the blowup $\pi_2 : X_2 \rightarrow X_1$. We denote by $\pi_1 : X_1 \rightarrow X_0$ the blowup of C_0 at the points p_i .

We can write $\zeta = \pi_2^*(\xi) - \sum_j \alpha_j F_j$, where $\alpha_j \geq 0$ and ξ is a movable class on X_1 . Since D_j are disjoint, by intersecting the equations $\zeta^2 = \zeta.c_1(X_2) = 0$ with F_j , we find as in [40] that either $\alpha_j = 0$ or

$$\xi.D_j = \alpha_j c_1(X_1).D_j = \alpha_j (2g_j - 2).$$

If $\alpha_j = 0$ then

$$\xi.D_j = \zeta.D'_j \geq 0 = \alpha_j c_1(X_1).D_j,$$

where $D'_j \subset F_j$ is a section whose pushforward is D_j . If $\alpha_j \neq 0$ then $\xi.D_j = c_1(X_1).D_j$. Therefore,

$$\begin{aligned} 0 \geq \zeta.c_2(X_2) &= (\pi_2^*(\xi) - \sum_j \alpha_j F_j).(\pi_2^*c_2(X_1) + \sum_j (\pi_2^*D_j - \pi_2^*c_1(X_j).F_j)) \\ &= \xi.c_2(X_1) + \sum_j (\xi.D_j - \alpha_j c_1(X_1).D_j). \end{aligned}$$

Since each term $\xi.D_j - \alpha_j c_1(X_1).D_j$ is non-negative, we find that $\xi.c_2(X_1) \leq 0$. Because $c_2(X_1) = \pi_1^*c_2(X_0)$, we then get that $(\pi_1)_*(\xi).c_2(X_0) \leq 0$. Because $(\pi_1)_*(\xi)$ is movable in X_0 , from the assumption on $c_2(X_0)$ we obtain $(\pi_1)_*(\xi) = 0$. From this, it easy follows that ξ and then ζ are 0.

2) The proof is similar to that of 1). The difference is now that here for each j , either $\alpha_j = 0$ or

$$\xi.D_j - \alpha_j c_1(X_1).D_j = \frac{\alpha_j}{2} [(2g_j - 2) - c_1(X_1).D_j].$$

In the first case

$$\xi.D_j - \alpha_j c_1(X_1).D_j = \xi.D_j = \zeta.D'_j \geq 0,$$

where $D'_j \subset F_j$ is a section. In the second case, by the assumption $(2g_j - 2) - c_1(X_1).D_j \geq 0$, we also have $\xi.D_j - \alpha_j c_1(X_1).D_j \geq 0$.

Hence,

$$0 \geq - \sum_j (\xi.D_j - \alpha_j c_1(X_1).D_j) \geq \xi.c_2(X_1).$$

Then we can proceed as before. \square

Proof of Theorem 6. Let F be the exceptional divisor of the blowup π . Let ζ be a nef class on X . Then we can write $\zeta = \pi^*(\xi) - \alpha F$ for some $\alpha \geq 0$ and for some movable class $\xi = \pi_*(\zeta)$ on Y .

Assume that $\zeta^2 = 0$. Then,

$$0 = \zeta^2 = (\pi^*(\xi) - \alpha F)^2 = \pi^*(\xi^2) + \alpha^2 F^2.$$

Here we used that $\pi^*(\xi).F = 0$. Because the classes of $\pi^*(\xi^2)$ and F^2 are linearly independent in the $(2, 2)$ cohomology group of X , from the above we have that $\alpha = 0$. Then it follows that ξ is nef on Y , and $\xi^2 = 0$. Moreover, since $c_1(X) = \pi^*c_1(Y) - 2F$ and $c_2(X) = \pi^*c_2(Y)$ (see Chapter 4 in [25]), we have

$$\begin{aligned} \xi.c_1(Y) &= \pi_*(\pi^*(\xi).\pi^*(c_1(Y))) = \pi_*(\pi^*(\xi).(\pi^*c_1(Y) - 2F)) = \pi_*(\zeta.c_1(X)), \\ \xi.c_1(Y)^2 &= \zeta.c_1(X)^2, \\ \xi.c_2(Y) &= \zeta.c_2(X). \end{aligned}$$

Then, it follows easily that if Y satisfies one of the Conditions A and B, then X also satisfies the same Condition. \square

Proof of Theorem 7. We will show that if Y satisfies Condition A then X also satisfies Condition A. The proof for Condition B is similar.

Let ζ be a nef class on X . We need to show that if

$$\begin{aligned} \zeta^2 &= 0, \\ \zeta.c_1(X)^2 &\geq 0, \\ \zeta.c_2(X) &\leq 0, \end{aligned}$$

then $\zeta \in \mathbb{R}.NS_{\mathbb{Q}}(X)$.

Let F be the exceptional divisor of the blowup $\pi : X \rightarrow Y$. We can write $\zeta = \pi^*(\xi) - \alpha F$ for some $\alpha \geq 0$. We also have (see Section 2)

$$\begin{aligned} c_1(X) &= \pi^*c_1(Y) - F, \\ c_2(X) &= \pi^*c_2(Y) + \pi^*C - \pi^*c_1(Y).F, \\ \pi_*(F.F) &= -C. \end{aligned}$$

We first consider the case $\alpha = 0$. Then, ξ is nef on Y and moreover $\xi^2 = 0$. We have in this case

$$\begin{aligned}\pi_*(\zeta.c_1(X)) &= \pi_*(\pi^*(\xi).(\pi^*c_1(Y) - F)) = \xi.c_1(Y), \\ \zeta.c_1(X)^2 &= \pi^*(\xi).(\pi^*c_1(Y) - F)^2 = \pi^*(\xi).(\pi^*c_1(Y)^2 - 2\pi^*c_1(Y).F + F^2) \\ &= \xi.c_1(Y)^2 - \xi.C, \\ \zeta.c_2(X) &= \pi^*(\xi).(\pi^*c_2(Y) + \pi^*C - \pi^*c_1(Y).F) \\ &= \xi.c_2(Y) + \xi.C.\end{aligned}$$

Since ξ is nef and C is an effective curve, we have $\xi.C \geq 0$. Therefore, from the assumptions $\zeta.c_1(X)^2 \geq 0$ and $\zeta.c_2(Y) \leq 0$ we obtain

$$\begin{aligned}\xi^2 &= 0, \\ \xi.c_1(Y)^2 &= \zeta.c_1(X)^2 + \xi.C \geq 0, \\ \xi.c_2(Y) &= \zeta.c_2(X) - \xi.C \leq 0.\end{aligned}$$

Since Y satisfies Condition A by assumption, it follows that $\xi \in \mathbb{R}.NS_{\mathbb{Q}}(Y)$. Then $\zeta = \pi^*(\xi) \in \mathbb{R}.NS_{\mathbb{Q}}(X)$. Hence, X also satisfies Condition A.

Now we show that under the assumptions of Theorem 6, then actually α must be 0. Assume otherwise, i.e. that $\alpha > 0$, we will obtain a contradiction. We recall that $\gamma = c_1(Y).C + 2g - 2$. From the assumption that $\zeta^2 = 0$ we have

$$\begin{aligned}0 &= \zeta^2.F = (\pi^*(\xi) - \alpha F)^2.F \\ &= \pi^*(\xi^2).F - 2\alpha\pi^*(\xi).F^2 + F^3 \\ &= \alpha\xi.C - 2\alpha^2.\gamma.\end{aligned}$$

In the fourth equality we used the results in Section 2. The assumption that $\alpha > 0$ implies that

$$\xi.C = \alpha.\gamma/2.$$

We now proceed corresponding to parts 1), 2) and 3) of the theorem.

1) In this case $c_1(Y).C$ is an odd number and $N_{C/Y}$ is decomposable. We have a SES of vector bundles over C :

$$0 \rightarrow T_C \rightarrow T_Y|_C \rightarrow N_{C/Y} \rightarrow 0.$$

From this, it follows that

$$c_1(N_{C/Y}) = c_1(Y).C + 2g - 2 = \gamma.$$

Recall that F is the exceptional divisor of the blowup π . Then $F = \mathbb{P}(N_{C/X}) \rightarrow C$ is a ruled surface over C . Hence, (see Proposition 2.8 in Chapter 5 in [26]) there is a line bundle \mathcal{M} over C such that $\mathcal{E} = N_{C/Y} \otimes \mathcal{M}$ is normalized, in the sense that $H^0(\mathcal{E}) \neq 0$, but for every line bundle \mathcal{L} with $c_1(\mathcal{L}) < 0$ then $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$.

Let f be a fiber of the fibration $F \rightarrow C$. Then, (see Proposition 2.9 in Chapter 5 in [26]), there is a so-called zero section $C_0 \subset F$ with the following properties:

$$\begin{aligned}\tau := C_0.C_0 &= c_1(\mathcal{E}), \\ C_0.f &= 1.\end{aligned}$$

Because $N_{C/Y}$ is decomposable, \mathcal{E} is also decomposable. By part a) of Theorem 2.12 in Section 5 in [26], $c_1(\mathcal{E}) \leq 0$. Moreover, from

$$c_1(\mathcal{E}) = c_1(\mathcal{N}_{C/Y}) + 2c_1(\mathcal{M}) = c_1(Y).C + 2g - 2 + 2c_1(\mathcal{M}),$$

and the assumption that $c_1(Y).C$ is an odd number, we get that $c_1(\mathcal{E}) < 0$. Hence $\tau < 0$.

From the results in Section 2, we have

$$C_0 = -F.F + \frac{1}{2}(\tau + \gamma)f.$$

Now we obtain the desired contradiction. Since ζ is nef and C_0 is an effective curve, we have $\zeta.C_0 \geq 0$. Hence,

$$\begin{aligned} 0 &\leq (\pi^*(\xi) - \alpha F).(-F.F + \frac{1}{2}(\tau + \gamma)f) \\ &= \xi.\pi_*(-F.F) + \alpha F.F.F - \frac{1}{2}\alpha(\tau + \gamma)F.f \\ &= \xi.C - \alpha\gamma + \frac{1}{2}\alpha(\tau + \gamma) = \frac{\alpha\tau}{2} < 0. \end{aligned}$$

In the above we used that $\pi_*(-F.F) = \pi_*(C_0) = C$ (see for example Lemma 4 in [40]), $F.f = -1$, $F.F.F = -\gamma$, $\xi.C = \alpha\gamma/2$, $\alpha > 0$ and $\tau = C_0.C_0 < 0$.

2) In this case, $\gamma < 0$ and C is not the only effective curve in its cohomology class. Let D be another curve in the cohomology class of C . Since C is irreducible, we can assume that C is not contained in the support of D . Then $\pi^*(D)$ is an effective curve in X . Since ζ is nef, we obtain a contradiction

$$0 \leq \pi^*(D).\zeta = D.\pi_*(\zeta) = D.\xi = C.\xi = \alpha\gamma/2 < 0.$$

3) In this case, there is an irreducible hypersurface $S \subset Y$ such that $2\kappa < \mu\gamma$. Here $\kappa = S.C$ and μ is the multiplicity of C in S . We now construct an effective curve $C_0 \subset F$ and use it to derive a contradiction.

The strict transform \tilde{S} of S is given by $\tilde{S} = \pi^*(S) - \mu F$, and is an irreducible hypersurface of X . Since \tilde{S} and F are different irreducible hypersurfaces, their intersection $C_0 = \tilde{S}.F = (\pi^*(S) - \mu F).F$ is an effective curve of F . We now compute the numbers $C_0.C_0$ and $C_0.M$. We have

$$\begin{aligned} C_0.C_0 &= \tilde{S}|_F.\tilde{S}|_F = \tilde{S}.\tilde{S}.F \\ &= (\pi^*(S) - \mu F).(\pi^*(S) - \mu F).F = -2\mu\pi^*(S).F.F + \mu^2 F.F.F \\ &= 2\mu S.C - \mu^2\gamma = 2\mu\kappa - \mu^2\gamma. \end{aligned}$$

Denote by $\tau = C_0.C_0$ and $\mu_0 = C_0.M$. Note that $\mu_0 \neq 0$, otherwise we have C_0 is a multiplicity of M , and hence $\pi_*(C_0) = 0$. But from the definition of C_0 we can see that $\pi_*(C_0) = \mu C \neq 0$. Then by the computations in Section 2, we have

$$F.F = -\frac{1}{\mu_0}C_0 + \frac{1}{2}\left(\frac{\tau}{\mu_0} + \gamma\right)M.$$

Pushforward this by the map π , using that $\pi_*(F.F) = -C$ and $\pi_*(C_0) = \mu C$ we have that $\mu_0 = \mu$.

From the above computation $\tau = 2\mu\kappa - \mu^2\gamma$, we obtain

$$F.C_0 = \frac{1}{2}\left(\gamma\mu - \frac{\tau}{\mu}\right) = \gamma\mu - \kappa.$$

Because ζ is nef, it follows that

$$\begin{aligned} 0 &\leq \zeta.C_0 = (\pi^*(\xi) - \alpha F).C_0 = \mu\xi.C - \frac{\alpha}{2}(\gamma\mu - \frac{\tau}{\mu}), \\ &= \frac{\alpha}{2}\gamma\mu - \frac{\alpha}{2}(\gamma\mu - \frac{\tau}{\mu}) = \frac{\alpha}{2}\frac{\tau}{\mu} = \alpha(\kappa - \frac{1}{2}\gamma\mu). \end{aligned}$$

This contradicts the assumptions that $2\kappa < \gamma\mu$ and $\alpha > 0$. \square

Proof of Theorem 8. Let ζ be a nef class on X . We need to show that if

$$\begin{aligned} \zeta^2 &= 0, \\ \zeta.c_1(X) &= 0, \\ \zeta.c_2(X) &\leq 0, \end{aligned}$$

then $\zeta \in \mathbb{R}.NS_{\mathbb{Q}}(X)$.

Let F be the exceptional divisor of the blowup $\pi : X \rightarrow Y$. We can write $\zeta = \pi^*(\xi) - \alpha F$ for some $\alpha \geq 0$. As in the proof of Theorem 7, it suffices to show that $\alpha = 0$. We assume otherwise that $\alpha > 0$. Let $f \subset F$ be a fiber of the projection $F \rightarrow C$. We have

$$\begin{aligned} 0 &= \zeta.\zeta = (\pi^*(\xi) - \alpha F).(\pi^*(\xi) - \alpha F) \\ &= \pi^*(\xi.\xi) - 2\alpha\pi^*(\xi).F + \alpha^2 F.F, \\ 0 &= \zeta.c_1(X) = (\pi^*(\xi) - \alpha F).(\pi^*c_1(Y) - F) \\ &= \pi^*(\xi.c_1(Y)) - \pi^*(\xi).F - \pi^*c_1(Y).F + \alpha F^2. \end{aligned}$$

Intersecting both of these equations with F , using $F.F.F = -\gamma$ and $\pi_*(F.F) = -C$, we obtain

$$\begin{aligned} 2\alpha\xi.C - \alpha^2\gamma &= 0, \\ \alpha c_1(Y).C + \xi.C - \alpha\gamma &= 0. \end{aligned}$$

Then we must have $\alpha = 0$. Otherwise, dividing 2α from the first equation we have that $\xi.C = \alpha\gamma/2$. Substituting this into the second equation and dividing by α we get $2c_1(Y).C = \gamma$. Hence $c_1(Y).C = 2g - 2$, which is a contradiction. \square

4. EXAMPLES

4.1. The case $X_0 = \mathbb{P}^2 \times \mathbb{P}^1$. The Picard number of X_0 is 2. By Künneth's formula, $H^{1,1}(X_0)$ is generated by the classes of $\mathbb{P}^2 \times \{pt\}$ and $\mathbb{P}^1 \times \mathbb{P}^1$ (here $\{pt\}$ means a point). The intersection on $H^{1,1}(X_0)$ is

$$\begin{aligned} \mathbb{P}^2 \times \{pt\}.\mathbb{P}^2 \times \{pt\} &= 0, \\ \mathbb{P}^2 \times \{pt\}.\mathbb{P}^1 \times \mathbb{P}^1 &= \mathbb{P}^1 \times \{pt\}, \\ \mathbb{P}^1 \times \mathbb{P}^1.\mathbb{P}^1 \times \mathbb{P}^1 &= \{pt\} \times \mathbb{P}^1. \end{aligned}$$

By Künneth's formula again, $H^{2,2}(X_0)$ is generated by $\mathbb{P}^1 \times \{pt\}$ and $\{pt\} \times \mathbb{P}^1$. The pairing between $H^{1,1}(X_0)$ and $H^{2,2}(X_0)$ is given by

$$\begin{aligned} \mathbb{P}^2 \times \{pt\}.\mathbb{P}^1 \times \{pt\} &= 0, \\ \mathbb{P}^2 \times \{pt\}.\{pt\} \times \mathbb{P}^1 &= 1, \\ \mathbb{P}^1 \times \mathbb{P}^1.\mathbb{P}^1 \times \{pt\} &= 1, \\ \mathbb{P}^1 \times \mathbb{P}^1.\{pt\} \times \mathbb{P}^1 &= 0. \end{aligned}$$

By Whitney's formula, we have

$$\begin{aligned} c_1(X_0) &= 2\mathbb{P}^2 \times \{pt\} + 3\mathbb{P}^1 \times \mathbb{P}^1, \\ c_2(X_0) &= 6\mathbb{P}^1 \times \{pt\} + 3\{pt\} \times \mathbb{P}^1. \end{aligned}$$

Therefore, we can check that X_0 satisfies all the conditions of Theorems 5, 6, 7 and 8. In particular, if $D_1, \dots, D_n \subset X_0$ are pairwise disjoint smooth curves, and $\pi_1 : X_1 \rightarrow X_0$ is the blowup at D_1, \dots, D_n , then for any automorphism f of X_1 we have $\lambda_1(f) = \lambda_2(f)$. However, X_0 does not satisfy the conditions of Theorem 4, its Picard number is $2 > 1$. For an appropriate choice of curves D_1, \dots, D_n , the threefold X_1 has automorphisms of positive entropy. In fact, there is a rational surface S obtained from \mathbb{P}^2 by blowing up distinct points $p_1, \dots, p_n \in \mathbb{P}^2$ such that S has an automorphism of positive entropy. If we choose $D_j = p_j \times \mathbb{P}^1$, then D_j are smooth rational curves which are disjoint, and X_1 has an automorphism of positive entropy.

4.2. The case $X_0 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. This case is very similar to the case $X_0 = \mathbb{P}^2 \times \mathbb{P}^1$ above. The readers can easily redo all the (analogs of) computations and constructions in the previous section.

4.3. The case $X_0 = \text{a complete intersection in } \mathbb{P}^N$. Let X_0 be a smooth projective threefold which is a complete intersection in \mathbb{P}^N . This means that X_0 is the intersection of smooth hypersurfaces D_1, \dots, D_{N-3} of \mathbb{P}^N . By Lefschetz's hyperplane theorem, X_0 has Picard number 1. We now show that X_0 satisfies the conditions of Theorem 5.

Lemma 7. *Let ζ be a non-zero movable class in X_0 . Then $\zeta \cdot c_2(X_0) > 0$.*

Proof. Let d_1, \dots, d_{N-3} be the degrees of V_1, \dots, V_{N-3} . Let h be the class of a hyperplane on X . The Chern classes of the normal bundle N_{X_0/\mathbb{P}^n} is given by the formula

$$c(N_{X_0/\mathbb{P}^n}) = \prod_{j=1}^{N-3} (1 + d_j h).$$

In particular,

$$\begin{aligned} c_1(N_{X_0/\mathbb{P}^n}) &= \left(\sum_j d_j \right) h, \\ c_2(N_{X_0/\mathbb{P}^n}) &= \left(\sum_{i < j} d_i d_j \right) h^2. \end{aligned}$$

From the exact sequence

$$0 \rightarrow T_{X_0} \rightarrow T_{\mathbb{P}^4}|_{X_0} \rightarrow N_{X_0/\mathbb{P}^3} \rightarrow 0,$$

and the splitting principle for Chern classes, it follows that

$$\begin{aligned} c_1(X_0) &= c_1(\mathbb{P}^n)|_{X_0} - c_1(N_{X_0/\mathbb{P}^n}) = ((n+1) - \sum_j d_j)h, \\ c_2(X_0) &= c_2(\mathbb{P}^n)|_{X_0} - c_2(N_{X_0/\mathbb{P}^n}) - c_1(X_0)c_1(N_{X_0/\mathbb{P}^n}) \\ &= \left(\frac{(n+1)n}{2} - \sum_{i < j} d_i d_j - (n+1) \sum_j d_j + \left(\sum_j d_j \right)^2 \right) h^2. \end{aligned}$$

We have

$$\begin{aligned} & \frac{(n+1)n}{2} - \sum_{i < j} d_i d_j - (n+1) \sum_j d_j + \left(\sum_j d_j \right)^2 \\ = & \left[\frac{n-4}{2(n-3)} \left(\sum_j d_j \right)^2 - \sum_{i < j} d_i d_j \right] + \left[\frac{n(n+1)}{2} + \frac{n-2}{2(n-3)} \left(\sum_j d_j \right)^2 - (n+1) \sum_j d_j \right]. \end{aligned}$$

By Cauchy-Schwarz inequality, the first bracket on the right hand side of the above expression is non-negative. We now show that the second bracket is positive. We define $x = \sum_j d_j$. Then x is a positive integer which is $\geq n-3$, and the second bracket is quadratic in x :

$$\frac{n(n+1)}{2} + \frac{n-2}{2(n-3)} \left(\sum_j d_j \right)^2 - (n+1) \sum_j d_j = \frac{n(n+1)}{2} - (n+1)x + \frac{(n-2)}{2(n-3)} x^2 =: g(x).$$

The critical point of g is $x_0 = (n+1)(n-3)/(n-2) < n$. Hence, to show that $g(x) > 0$ for all positive integer $x \geq n-3$, it suffices to show that $g(n-3), g(n-2), g(n-1), g(n) > 0$ for any positive integer $n \geq 4$. We now check this latter claim.

For $x = n-3$

$$g(n-3) = 6 > 0.$$

(Note that in this case all d_j are 1 and X_0 is no other than \mathbb{P}^3 .)

For $x = n-2$, using that $(n-2)^2 > (n-1)(n-3)$, we obtain

$$g(n-2) > \frac{n(n+1)}{2} - (n^2 - n - 2) + \frac{(n-2)(n-1)}{2} = 3 > 0.$$

For $x = n-1$, we have

$$g(n-1) = \frac{2(n-2)}{(n-3)} > 0.$$

For $x = n$, we have

$$g(n) = \frac{1}{(n-3)} > 0.$$

A movable class is in particular psef, i.e. can be represented by a positive closed current. Hence, if ζ is a non-zero movable class on X_0 then $\xi.c_2(X_0) > 0$. Hence, Theorem 5 can be applied for such a X_0 .

□

4.4. A generalization of Theorem 3. The proof of Theorem 6 shows that the conclusion is still valid in the following more general setting. Let $\pi_1 : X_1 \rightarrow X_0 = \mathbb{P}^3$ be the blowup at n points p_1, \dots, p_n . Let E_1, \dots, E_n be the exceptional divisors. Let $D_1, \dots, D_m \subset X_1$ be pairwise disjoint smooth curves. Let $X = X_2$ be the blowup of X_1 at D_1, \dots, D_m . We define

$$\gamma := \sum_j \deg(\pi_1)_*(D_j).$$

Assume that there is $\lambda > 0$ such that for any l :

$$\sum_j E_l \cdot D_j \leq \lambda,$$

and moreover

$$\frac{6 + \gamma}{\lambda} > \frac{11}{2}.$$

Moreover, assume that for any j

$$\left(\frac{1}{2} + \frac{1}{\lambda}\right)c_1(X_1).D_j \geq \frac{g_j - 1}{2},$$

where g_j is the genus of D_j .

5. A POSSIBLE APPLICATION TO THE UENO'S THREEFOLD

Let $E_{\sqrt{-1}}$ be an elliptic curve with an automorphism of order 4, which we denote by $\sqrt{-1}$. In [41], Ueno asked whether the quotient variety $E_{\sqrt{-1}}^3/\sqrt{-1}$ is rational. Campana [9] showed that the variety is rationally connected. Then, by a combination of the two papers [13] and [12], it follows that $E_{\sqrt{-1}}^3/\sqrt{-1}$ is rational. Previously, a similar construction, using instead an elliptic curve with an automorphism of order 3, has been shown to be rational (see [36]).

The automorphism $\sqrt{-1}$ on $E_{\sqrt{-1}}^3$ has 8 fixed points and $64 - 8$ points of period 2. Therefore, $E_{\sqrt{-1}}^3/\sqrt{-1}$ has $8 + 28 = 36$ singular points. Let X_4 be the minimal resolution of $E_{\sqrt{-1}}^3/\sqrt{-1}$, that is X_4 is the blowup of $E_{\sqrt{-1}}^3/\sqrt{-1}$ at the 36 singular points.

Since X_4 is birational equivalent to \mathbb{P}^3 , by the weak factorization theorem, X_4 can be obtained from \mathbb{P}^3 by a combination of smooth blowups and blowdowns. It is then natural to ask the following question:

Question 2. Can X_4 be obtained from \mathbb{P}^3 or $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by a finite composition of smooth blowups only?

This question is interesting in several aspects. First, the two dimensional analogue, that is the minimal resolution of $E_{\sqrt{-1}}^2/\sqrt{-1}$, has been shown to be a finite composition of point blowups starting from $\mathbb{P}^1 \times \mathbb{P}^1$ in [9]. Last, the final proof that X_4 is rational in [12] is rather abstract. Hence, if the answer to Question 2 is affirmative, it will give an explicit proof that X_4 is rational.

We note that the smooth threefold X_4 has automorphisms f coming from the complex torus $E_{\sqrt{-1}}^3$ with $\lambda_1(f) \neq \lambda_2(f)$. Therefore, from the discussion in the introduction of this paper, it is plausible to conclude that the answer to Question 2 is negative. The purpose of this section is to give more weight to this speculation.

We first show that if the answer for Question 2 is affirmative, then centers of the individual blowups must be smooth rational curves. In the below, for any quasi-projective variety Z we will denote by $\chi(Z)$ the Euler characteristic with compact support. For a smooth projective manifold Z , we denote by $\rho(Z)$ the Picard number of Z .

Theorem 10. *Let X_0 be any smooth projective threefold such that $\chi(X_0) = 2 + 2\rho(X_0)$ (for example, X_0 is \mathbb{P}^3 , $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$). Assume that X_4 can be obtained from the X_0 by a finite composition of smooth blowups. Then the curves which are centers of the blowups must be smooth rational curves.*

Proof. We divide the proofs into several steps.

Step 1. We claim that $\rho(X_4) = 45$. In fact, $E_{\sqrt{-1}}^3$ has Picard number 9. Also, the blowup $X_4 \rightarrow E_{\sqrt{-1}}^3/\sqrt{-1}$ has 36 exceptional divisors, one for each singular points. Hence the Picard number of X_4 is $9 + 36 = 45$.

Step 2. We claim that $\chi(X_4) = 92$. In fact, first we consider the quotient map $\sigma : E_{\sqrt{-1}}^3 \rightarrow E_{\sqrt{-1}}^3/\sqrt{-1}$. Let $A \subset E_{\sqrt{-1}}^3$ be the set of fixed points of $\sqrt{-1}$, and $B \subset E_{\sqrt{-1}}^3$ the set of points of period 2 of $\sqrt{-1}$. As mentioned before, the cardinals of $|A|$ and $|B|$ are 8 and 56, and the cardinals of $\sigma(A)$ and $\sigma(B)$ are 8 and 28. Since the map $\pi : E_{\sqrt{-1}}^3 - (A \cup B) \rightarrow E_{\sqrt{-1}}^3/\sqrt{-1} - (\sigma(A) \cup \sigma(B))$ is a 4 : 1 map, by the excision property we get

$$\begin{aligned} \chi(E_{\sqrt{-1}}^3/\sqrt{-1} - (\sigma(A) \cup \sigma(B))) &= \chi(E_{\sqrt{-1}}^3 - (A \cup B))/4 \\ &= (\chi(E_{\sqrt{-1}}^3) - \chi(A \cup B))/4 \\ &= (0 - 64)/4 = -16. \end{aligned}$$

Hence, by the excision property, we have $\chi(E_{\sqrt{-1}}^3/\sqrt{-1}) = -16 + 36 = 20$.

Next, we consider the blowup $\pi : X_4 \rightarrow E_{\sqrt{-1}}^3/\sqrt{-1}$. This map has 36 exceptional divisors, each is a \mathbb{P}^2 . Since $\chi(\mathbb{P}^2) = 3$ and the blowup map is 1 : 1 outside exceptional divisors, arguing as above we obtain

$$\chi(X_4) = (\chi(E_{\sqrt{-1}}^3/\sqrt{-1}) - 36\chi(pt)) + 36 \times \chi(\mathbb{P}^2) = 20 - 36 + 36 \times 3 = 92.$$

Here pt denotes a point.

Step 3: $\chi(X_4) = 2 + 2\rho(X_4)$. This follows from Steps 1 and 2.

Step 3. Let Z be a smooth projective threefold and $\pi : Z_1 \rightarrow Z$ a point blowup. Then $\chi(Z_1) = \chi(Z) + 2$. This follows easily from the properties of the blowup of a threefold.

Step 4. Let Z be a smooth projective threefold and $\pi : Z_1 \rightarrow Z$ a blowup at a smooth curve $C \subset Z$. Then $\chi(Z_1) \leq \chi(Z) + 2$, with equality if and only if C is a smooth rational curve. Again, this follows easily from the properties of the blowup of a threefold, and the fact that if C is not a smooth rational curve then $h^1(C) > 0$.

Step 5: final step. If X_4 is a finite composition of smooth blowups of X_0 , then the number of blowups needed is $\rho(X_4) - \rho(X_0) = 45 - \rho(X_0)$. From the previous steps we have

$$92 = \chi(X_4) \leq \chi(X_0) + 2(45 - \rho(X_0)) = 2 + 2\rho(X_0) + 90 - 2\rho(X_0) = 92.$$

Since equality occurs, it follows from Steps 3 and 4 that the centers of the individual blowups must be either a point or a smooth rational curve. \square

Now we show how Theorem 10 and Theorems 6, 7 and 8 almost give the proof that the answer to Question 2 is negative. In fact, let X_0 be \mathbb{P}^3 , $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then, X_0 satisfies Condition B, while X_4 does not satisfy Condition B. Assume that X_4 is a finite composition of smooth blowups starting from X_0 . Let $\pi_j : Z_{j+1} \rightarrow Z_j$ be an individual blowup in the sequence, where Z_j satisfies Condition B. If π_j is a point blowup then by Theorem 6, Z_{j+1} also satisfies Condition B. If π_j is the blowup of a smooth curve $C \subset Z_j$, then by Theorem 10, we have that C must be a smooth rational curve. If $c_1(Z_j).C \neq 2g - 2 = -2$, then by Theorem 8 we have that Z_{j+1} also satisfies Condition B. The remaining case is when $c_1(Z_j).C = -2$. But in this case, half of the conditions of part 2 of Theorem 7 is satisfied. The only condition that is missing is the condition that C is not the only effective curve in its cohomology class. Using part 1 of Theorem 6, we can also

show that if the normal vector bundle N_{C/Z_j} is not isomorphic to $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$, then Z_{j+1} also satisfies Condition B.

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SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 130-722, REPUBLIC OF KOREA

E-mail address: `truong@kias.re.kr`